# Propagators: An Introduction

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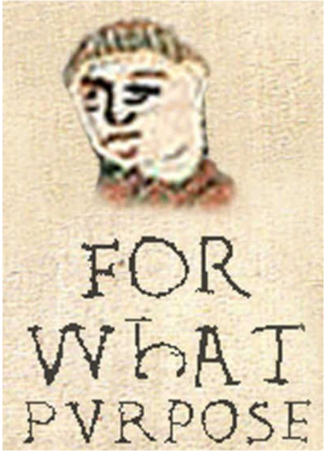
What?



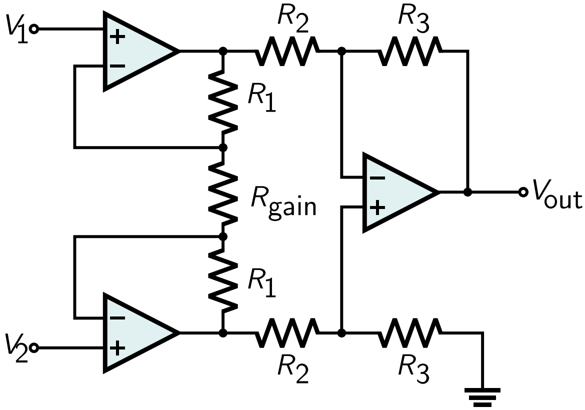

Why?

Beginnings as early as the 1970's at MIT

- Guy L. Steele Jr.
- Gerald J. Sussman
- Richard Stallman

More recently:

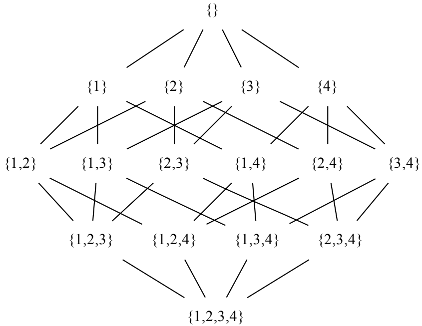
- Alexey Radul



```
(define (map f xs)
  (cond ((null? xs) '())
        (else (cons (f (car xs))
                    (map f (cdr xs))))))
```

And then

- Edward Kmett




$$x \leq y \implies f(x) \leq f(y)$$

They're related to many areas of research, including:

- Logic programming (particularly Datalog)
- Constraint solvers
- Conflict-Free Replicated Datatypes
- LVars
- Programming language theory
- And spreadsheets!

They have advantages:

- are extremely expressive
- lend themselves to parallel and distributed evaluation
- allow different strategies of problem-solving to cooperate

# Propagators

The *propagator model* is a model of computation
We model computations as *propagator networks*

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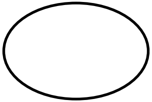
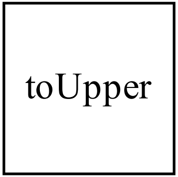
We model computations as *propagator networks*

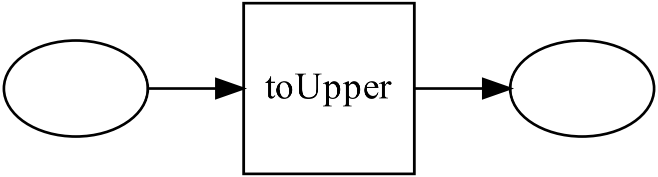
A propagator network comprises

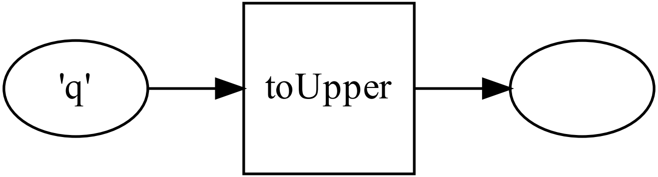
- cells
- propagators
- connections between cells and propagators

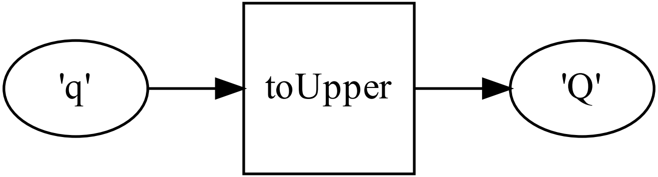


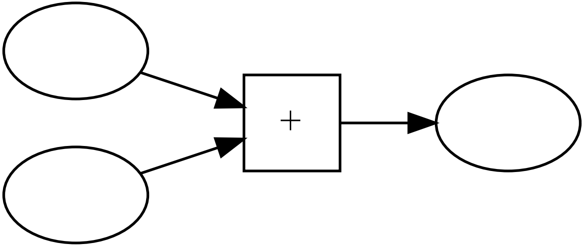
3

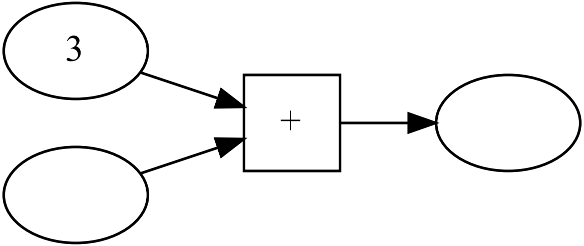

toUpper

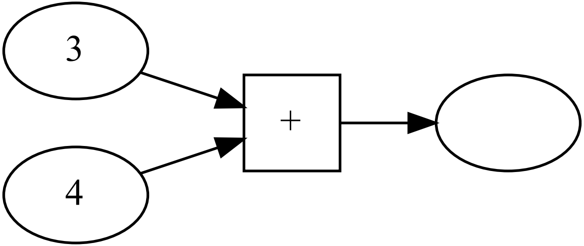

toUpper

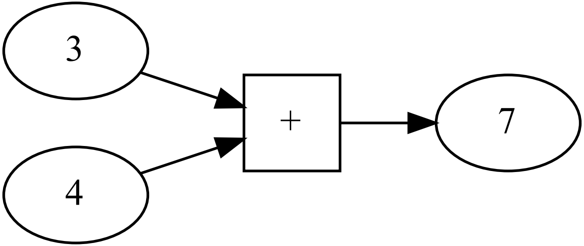

'q'
toUpper


'q'
toUpper
'Q'




3
+


3
4
+


3
4
+
7

$$z \leftarrow x + y$$

$$z = x + y$$

$$7 = x + 4$$

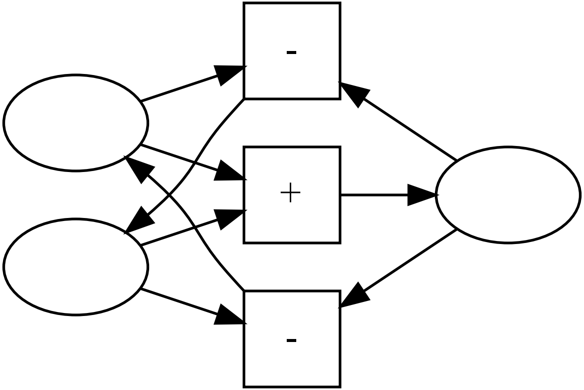
$$7 = 3 + 4$$

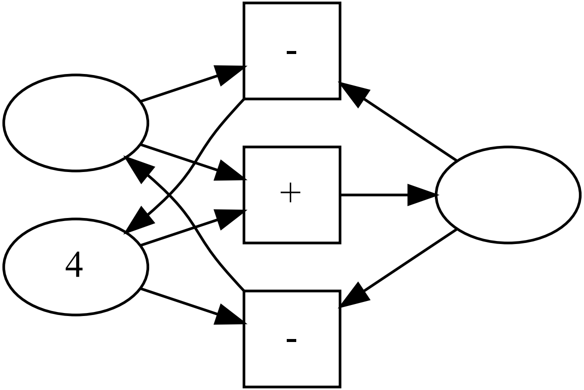
$$z = x + y$$

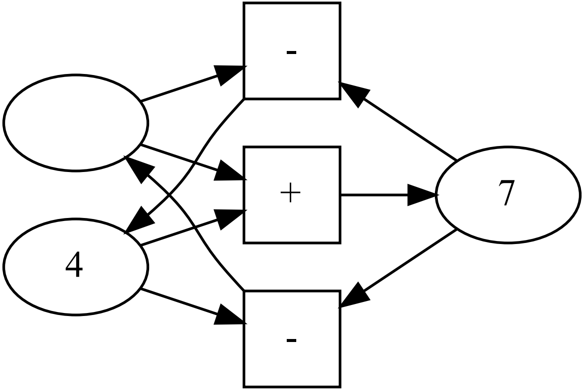
$$z \leftarrow x + y$$

$$x \leftarrow z - y$$

$$y \leftarrow z - x$$




-
+
4
-


-
+
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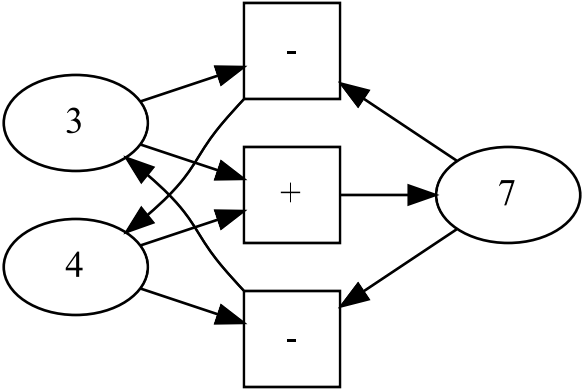
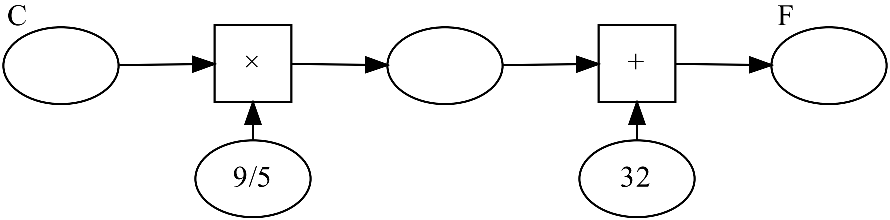
4
7



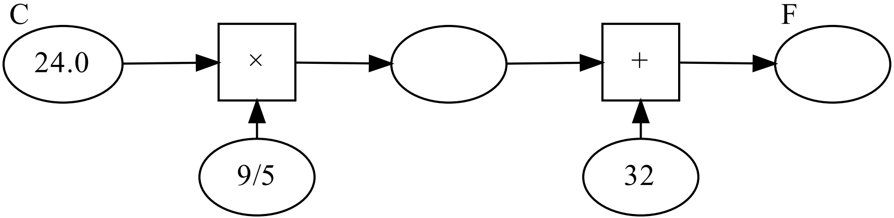
-
3
+
7
4
-

Propagators let us express bidirectional relationships!

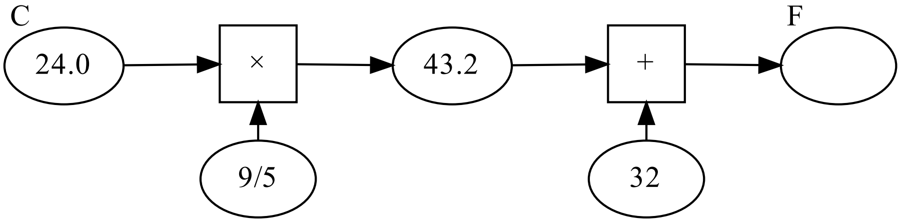
$$°F = °C \times \frac{9}{5} + 32$$



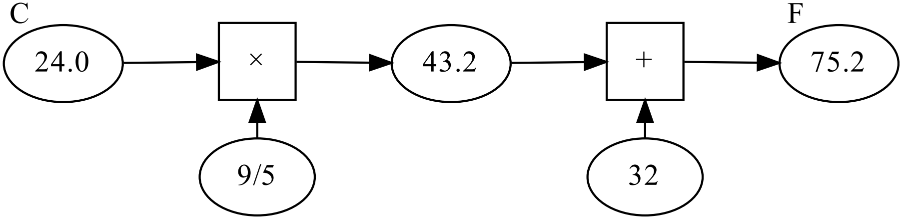
$$^{\circ}F = {}^{\circ}C \times \frac{9}{5} + 32$$



$$°F = °C \times \frac{9}{5} + 32$$

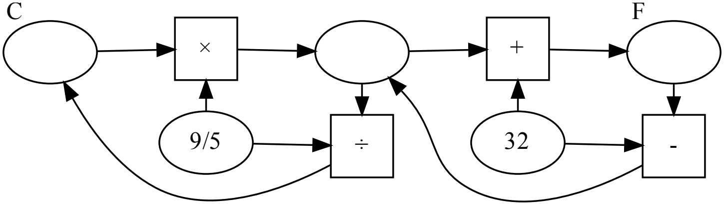


$$^\circ F = {^\circ C} \times \frac{9}{5} + 32$$



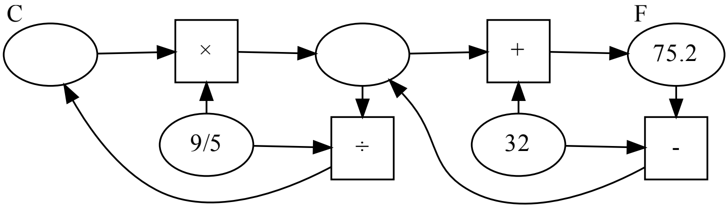
$$°F = °C \times \frac{9}{5} + 32$$

$$°C = (°F - 32) \div \frac{9}{5}$$



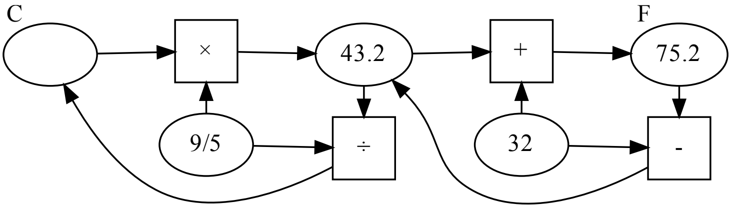
$$^\circ F = {^\circ C} \times \frac{9}{5} + 32$$

$$^\circ C = ({^\circ F} - 32) \div \frac{9}{5}$$



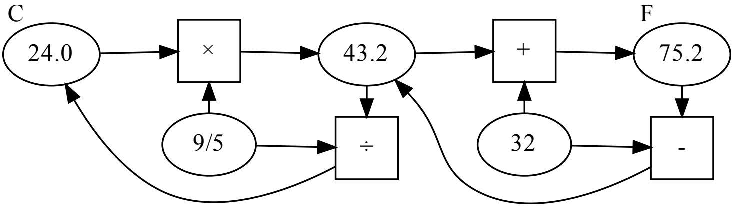
$$°F = °C \times \frac{9}{5} + 32$$

$$°C = (°F - 32) \div \frac{9}{5}$$



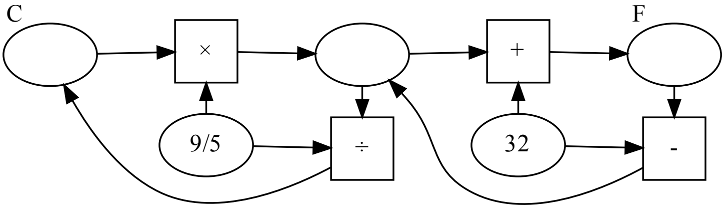
$$^\circ F = {}^\circ C \times \frac{9}{5} + 32$$

$$^\circ C = ({}^\circ F - 32) \div \frac{9}{5}$$



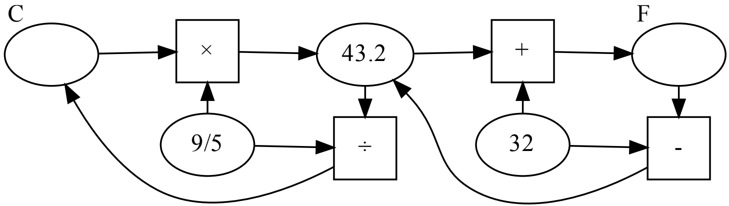
$$°F = °C \times \frac{9}{5} + 32$$

$$°C = (°F - 32) \div \frac{9}{5}$$



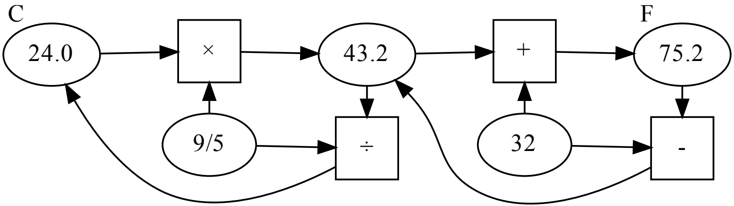
$$°F = °C \times \frac{9}{5} + 32$$

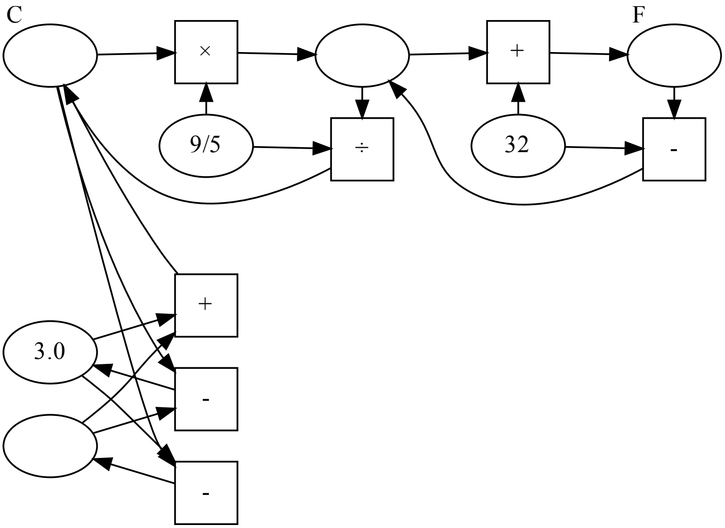
$$°C = (°F - 32) \div \frac{9}{5}$$

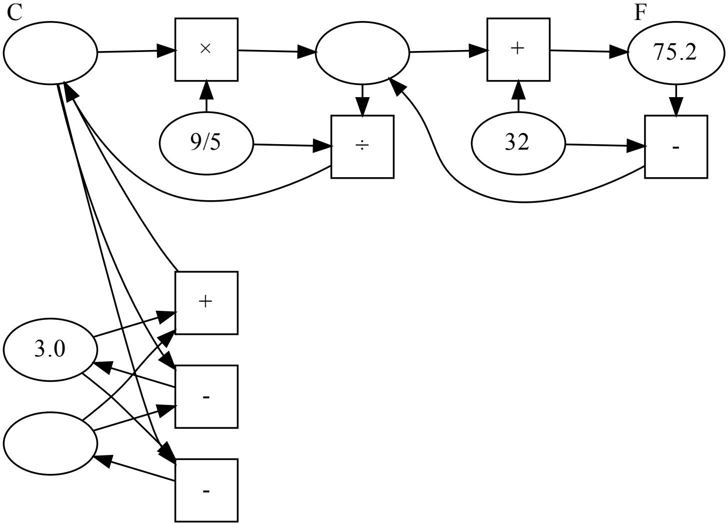


$$^\circ F = {}^\circ C \times \frac{9}{5} + 32$$

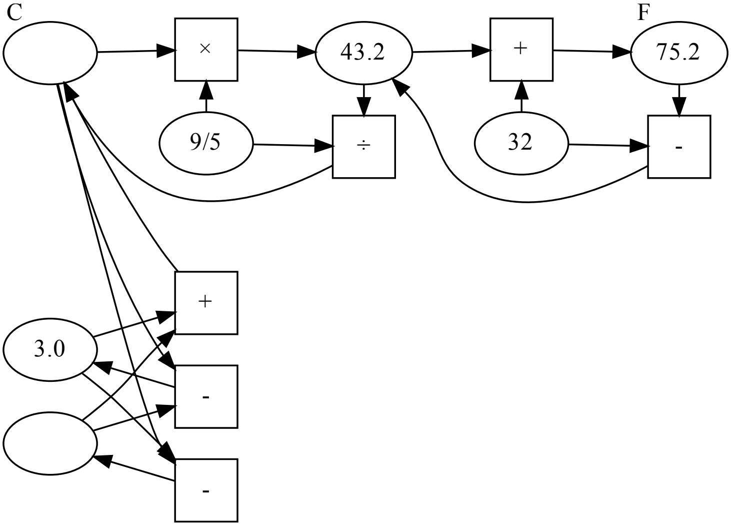
$$^\circ C = ({}^\circ F - 32) \div \frac{9}{5}$$



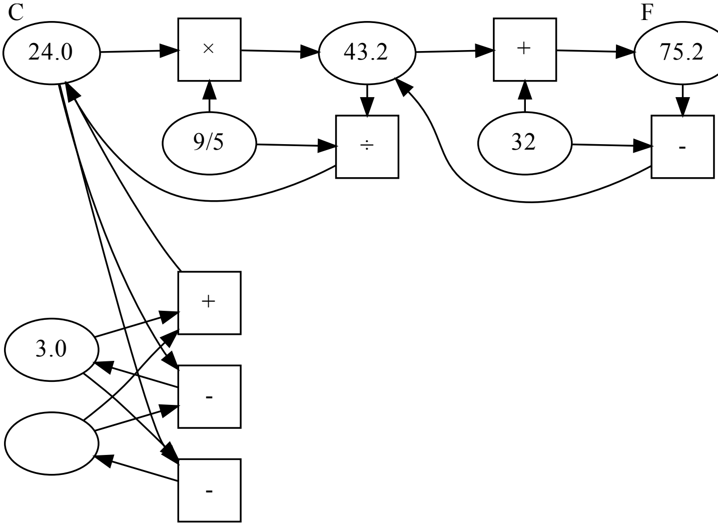

C
×
+
F
9/5
÷
32
-
+
3.0
-
-

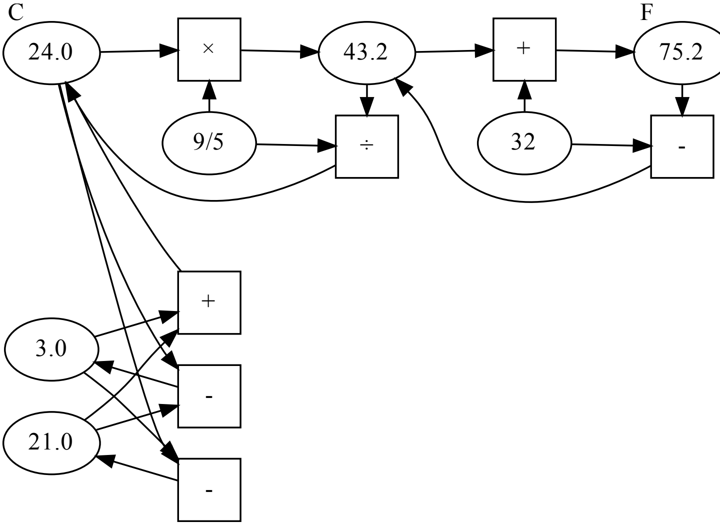


C
×
+
F
75.2
9/5
÷
32
-
+
3.0
-
-



C
×
43.2
+
F
75.2
9/5
÷
32
-
+
3.0
-
-


C
24.0
×
43.2
+
F
75.2
9/5
÷
32
-
+
3.0
-
-



C
24.0
×
43.2
+
F
75.2
9/5
÷
32
-
+
3.0
-
21.0
-

We can combine networks into larger networks!

?

Cells *accumulate information* about a value

| | | | |
|---|---|---|---|
| 3 | | | 2 |
| | 4 | 1 | |
| | 3 | 2 | |
| 4 | | | 1 |

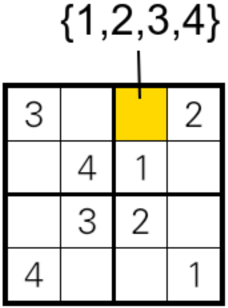
| | | | |
|---|---|---|---|
| 3 | | | 2 |
| | 4 | 1 | |
| | 3 | 2 | |
| 4 | | | 1 |

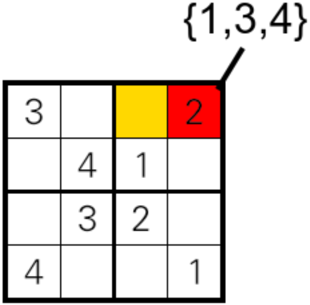
| | | | |
|---|---|---|---|
| 3 | | | 2 |
| | 4 | 1 | |
| | 3 | 2 | |
| 4 | | | 1 |

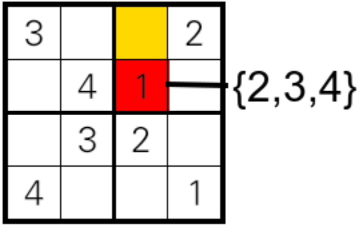
| | | | |
|---|---|---|---|
| 3 | | | 2 |
| | 4 | 1 | |
| | 3 | 2 | |
| 4 | | | 1 |

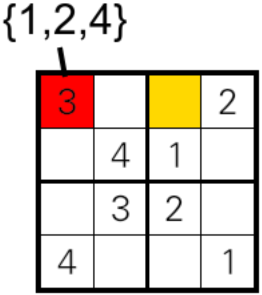
| | | | |
|---|---|---|---|
| 3 | | | 2 |
| | 4 | 1 | |
| | 3 | 2 | |
| 4 | | | 1 |

| | | | |
|---|---|---|---|
| 3 | | | 2 |
| | 4 | 1 | |
| | 3 | 2 | |
| 4 | | | 1 |


{1,2,3,4}
3 2
4 1
3 2
4 1


{1,3,4}
3
2
4
1
3
2
4
1


3
2
4
1
{2,3,4}
3
2
4
1


{1,2,4}
3 2
4 1
3 2
4 1

$$\{2,3,4\} \cap \{1,3,4\} \cap \{1,2,4\} \cap \{1,2,3,4\}$$

| | | | |
|---|---|---|---|
| 3 | | | 2 |
| | 4 | 1 | |
| | 3 | 2 | |
| 4 | | | 1 |

{4}

| | | | |
|---|---|---|---|
| 3 | | | 2 |
| | 4 | 1 | |
| | 3 | 2 | |
| 4 | | | 1 |

| | | | |
|---|---|---|---|
| 3 | | 4 | 2 |
| | 4 | 1 | |
| | 3 | 2 | |
| 4 | | | 1 |

Cells accumulate information in a *bounded join-semilattice*

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A bounded join-semilattice is:

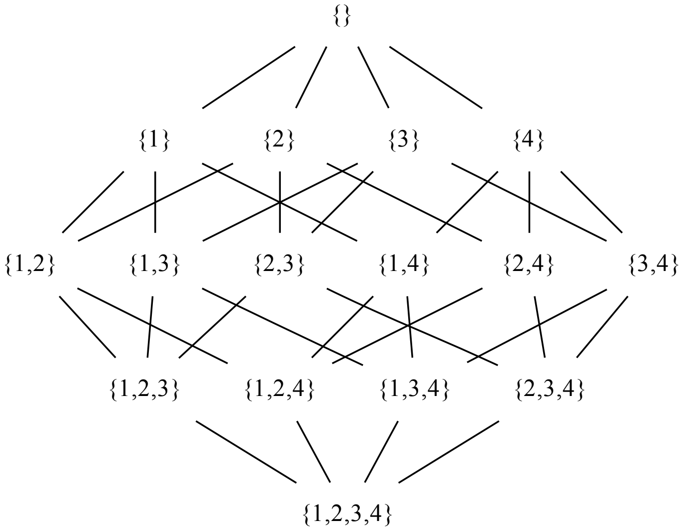
- A *partially ordered set*
- with a least element
- such that any subset of elements has a *least upper bound*

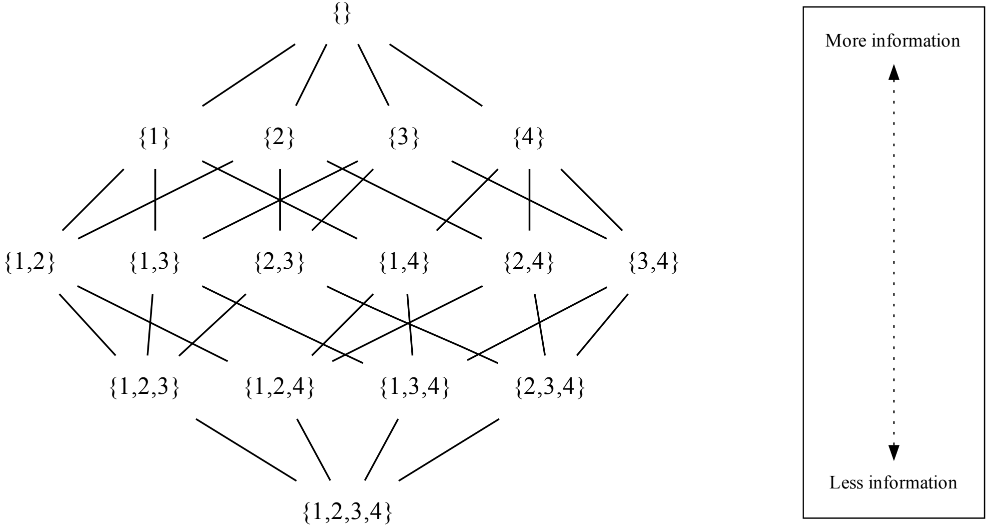
Cells accumulate information in a *bounded join-semilattice*

A bounded join-semilattice is:

- A *partially ordered set*
- with a least element
- such that any subset of elements has a *least upper bound*

"Least upper bound" is denoted as $\vee$ and is usually pronounced "join"


{}
{1}
{2}
{3}
{4}
{1,2}
{1,3}
{2,3}
{1,4}
{2,4}
{3,4}
{1,2,3}
{1,2,4}
{1,3,4}
{2,3,4}
{1,2,3,4}


{}
{1}
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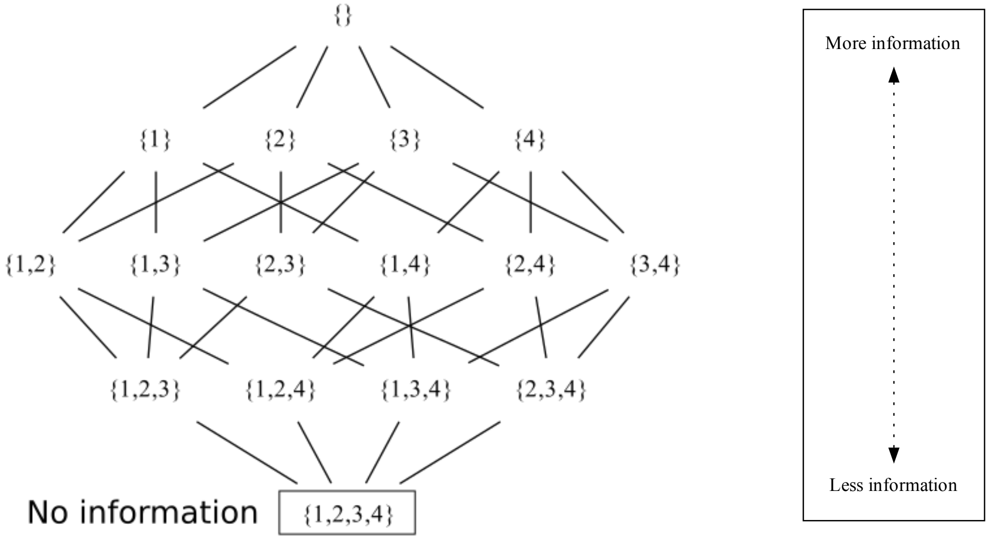
{1,3,4}
{2,3,4}
{1,2,3,4}
More information
Less information


{}
{1}
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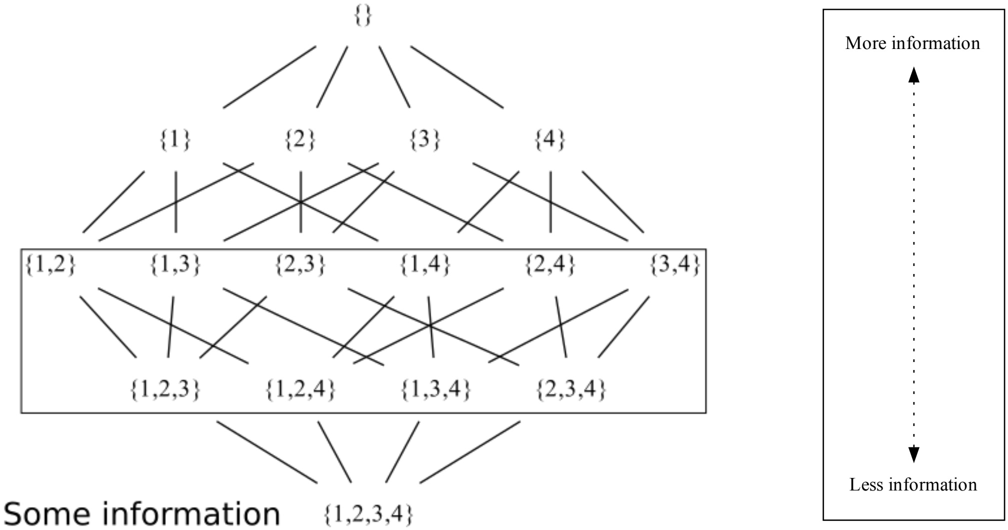
{2,3}
{1,4}
{2,4}
{3,4}
{1,2,3}
{1,2,4}
{1,3,4}
{2,3,4}
No information
{1,2,3,4}
More information
Less information


{}
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{2,4}
{3,4}
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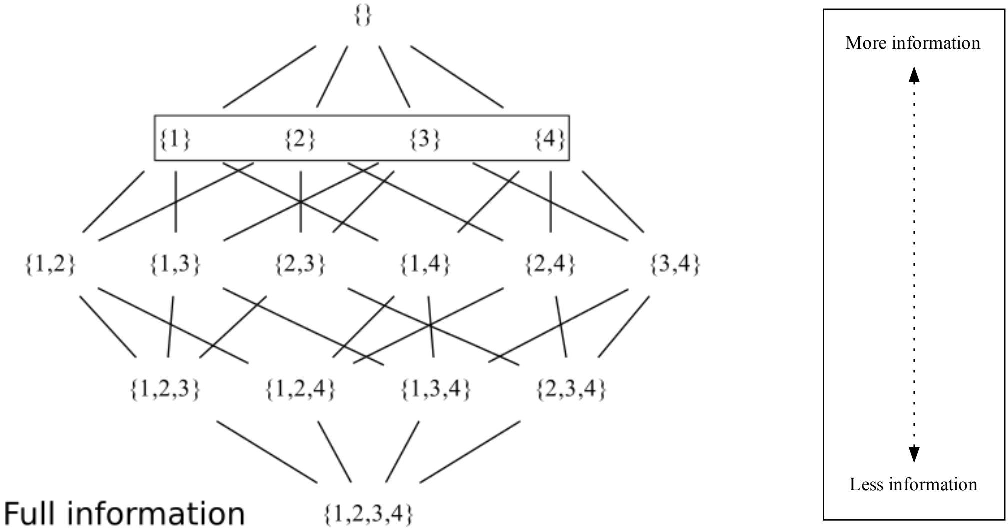
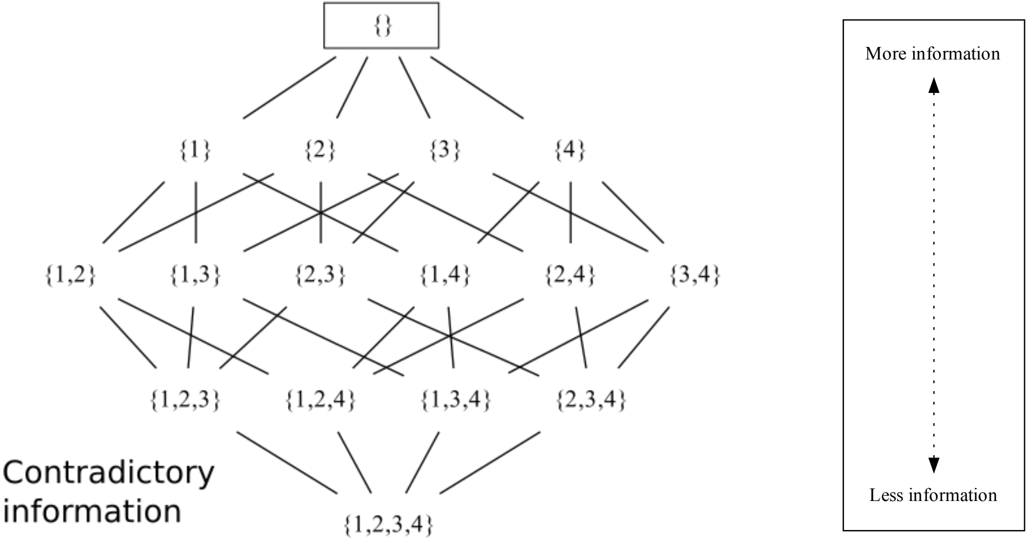
{1,2,4}
{1,3,4}
{2,3,4}
Some information
{1,2,3,4}
More information
Less information


{}
{1} {2} {3} {4}
{1,2} {1,3} {2,3} {1,4} {2,4} {3,4}
{1,2,3} {1,2,4} {1,3,4} {2,3,4}
Full information
{1,2,3,4}
More information
Less information


{}
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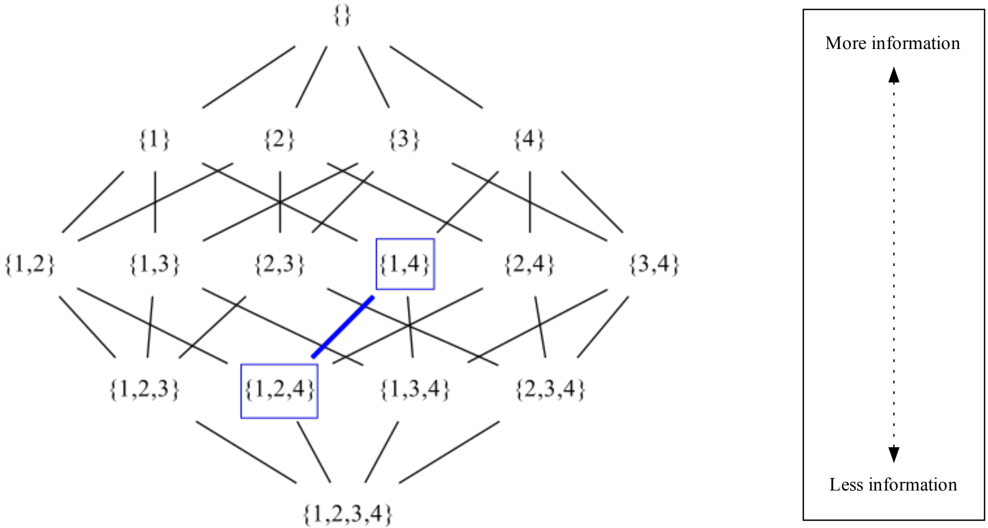
{1,2,3,4}
Contradictory information
More information
Less information


{}
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{1,2,3}
{1,2,4}
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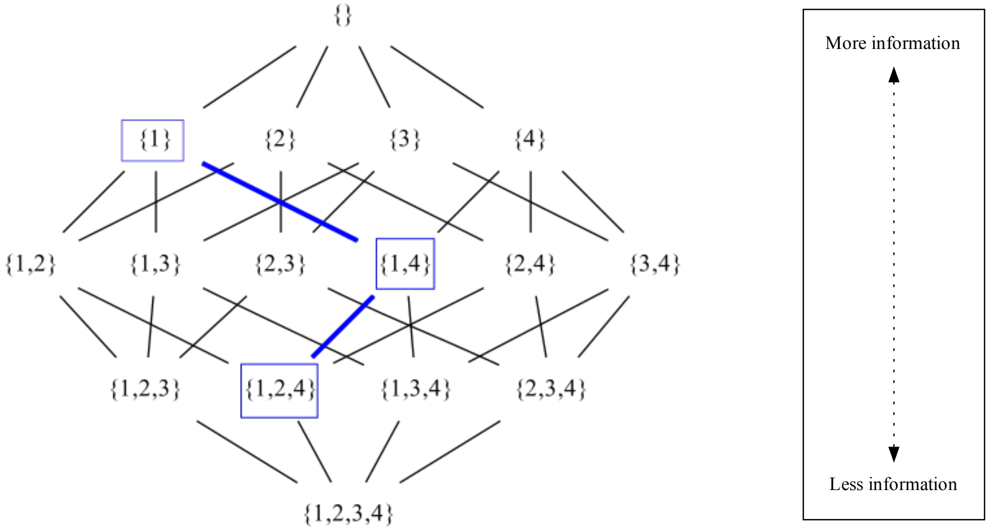
{2,3,4}
{1,2,3,4}
More information
Less information

{1,2,4} < {1,4}


{}
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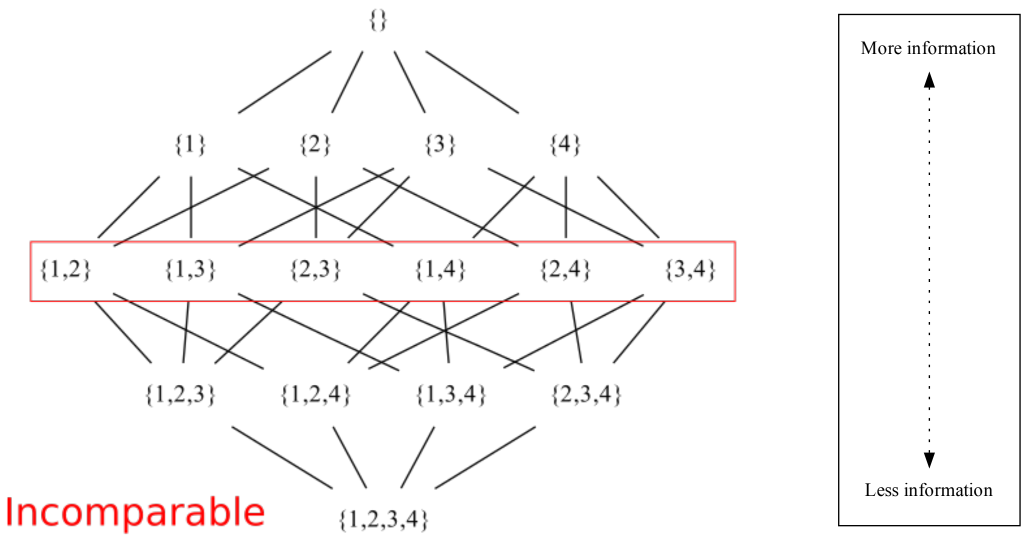
{1,2,3}
{1,2,4}
{1,3,4}
{2,3,4}
{1,2,3,4}
More information
Less information

$\{1,2,4\} < \{1,4\} < \{1\}$


{}
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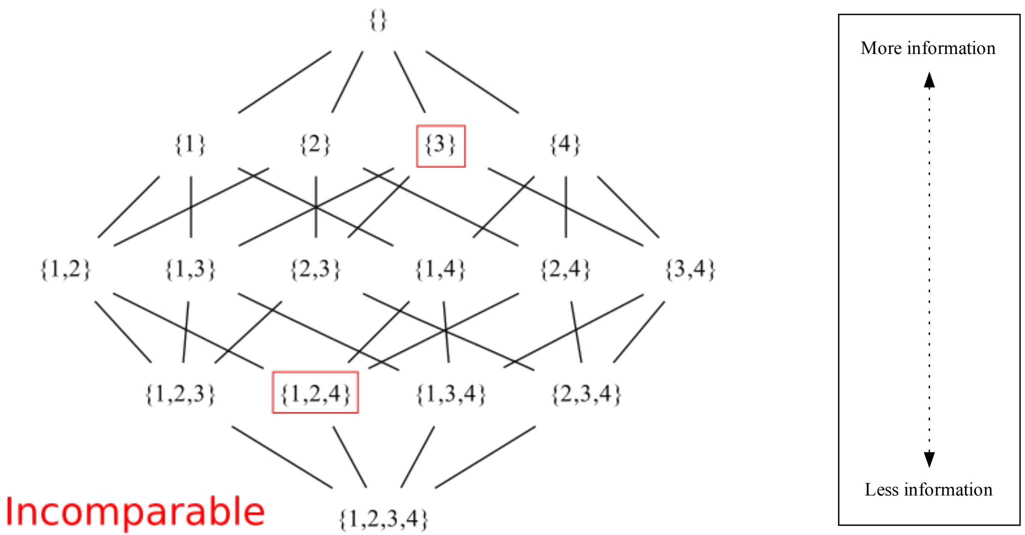
{2,3,4}
{1,2,3,4}
Incomparable
More information
Less information


{}
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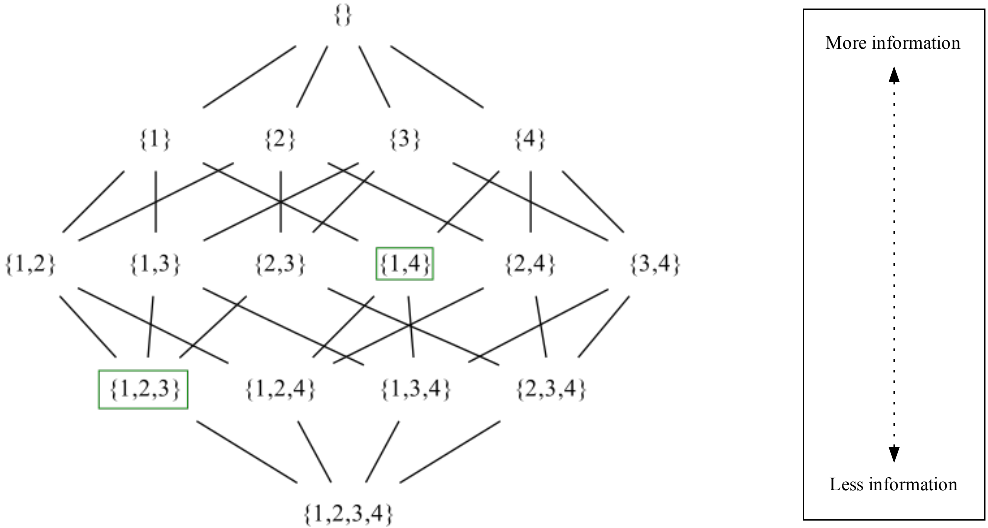
{1,2,3,4}
Incomparable
More information
Less information


{}
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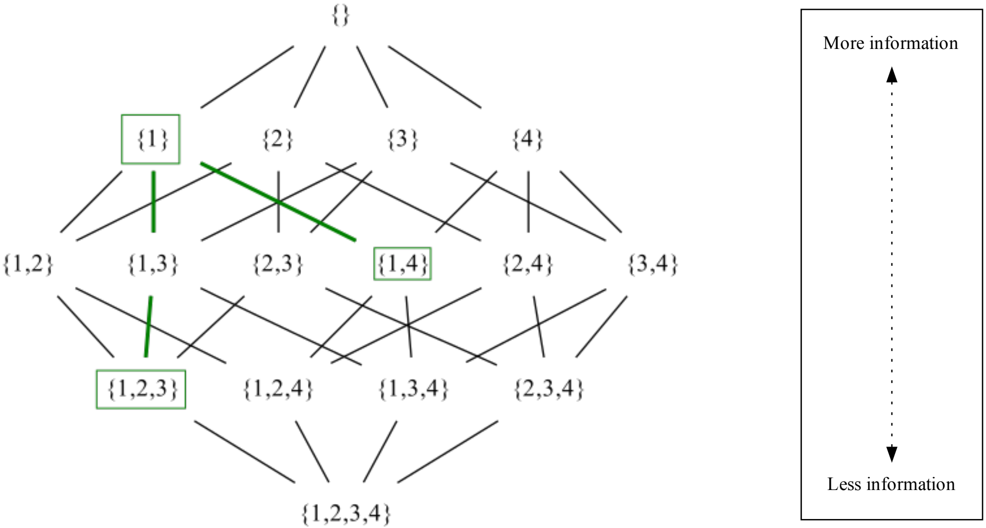
{1,2,3}
{1,2,4}
{1,3,4}
{2,3,4}
{1,2,3,4}
More information
Less information

{1,2,3} ∨ {1,4}


{}
{1}
{2}
{3}
{4}
{1,2}
{1,3}
{2,3}
{1,4}
{2,4}
{3,4}
{1,2,3}
{1,2,4}
{1,3,4}
{2,3,4}
{1,2,3,4}
More information
Less information

$\{1,2,3\} \vee \{1,4\} = \{1\}$

$\vee$ has useful algebraic properties. It is:

- A monoid
- that's commutative
- and idempotent

Left identity

$$\epsilon \vee x = x$$

Right identity

$$x \vee \epsilon = x$$

Associativity

$$(x \vee y) \vee z = x \vee (y \vee z)$$

Commutative

$$x \vee y = y \vee x$$

Idempotent

$$x \vee x = x$$

```haskell
class BoundedJoinSemilattice a where
  bottom :: a
  (\/) :: a -> a -> a
```

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data SudokuVal = One | Two | Three | Four
                 deriving (Eq, Ord, Show)
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data SudokuVal = One | Two | Three | Four
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newtype SudokuSet = S (Set SudokuVal)


instance BoundedJoinSemilattice SudokuSet where
  bottom     = S (Set.fromList [One, Two, Three, Four])
  S a \/ S b = S (Set.intersection a b)
```

We don't write values directly to cells
Instead we *join information in*

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This makes our propagators *monotone*, meaning that as the input cells gain information, the output cells gain information (or don't change)

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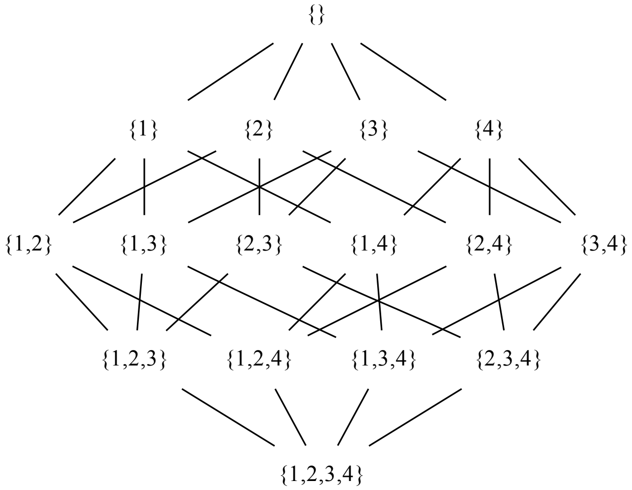
A function $f : A \to B$ where $A$ and $B$ are partially ordered sets is **monotone** if and only if, for all $x, y \in A.\ x \leq y \implies f(x) \leq f(y)$

# All our lattices so far have been fininte



Thanks to these properties:

- the bounded join-semilattice laws
- the finiteness of our lattice
- the monotonicity of our propagators

our propagator networks will yield with a deterministic answer, in finite time, regardless of parallelism and distribution

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See: Conflict Free Replicated Datatypes

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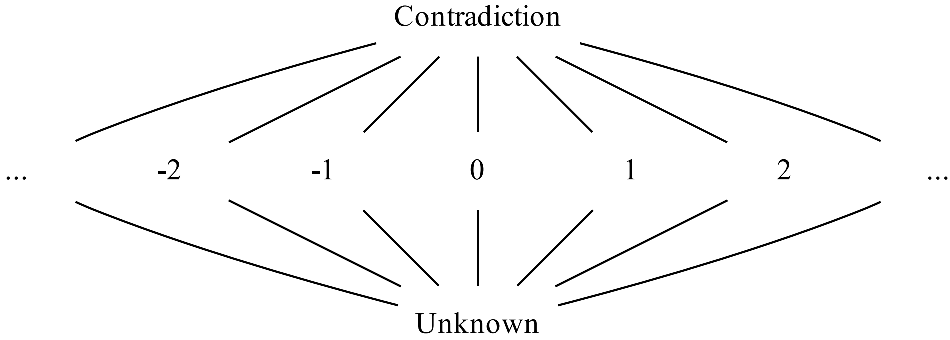
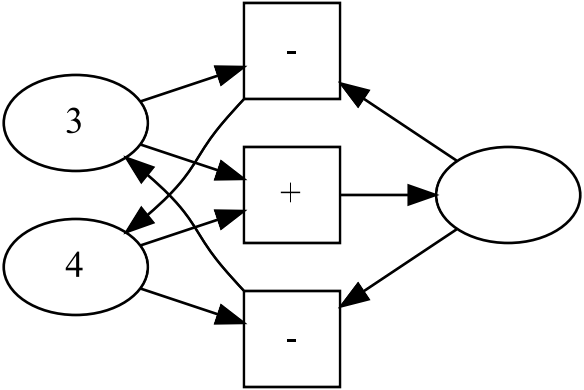
See: Conflict Free Replicated Datatypes

We can relax these constraints in a few different directions

Our lattices only need the *ascending chain condition*



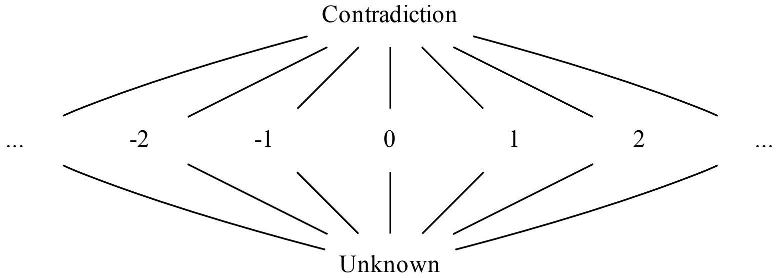
?

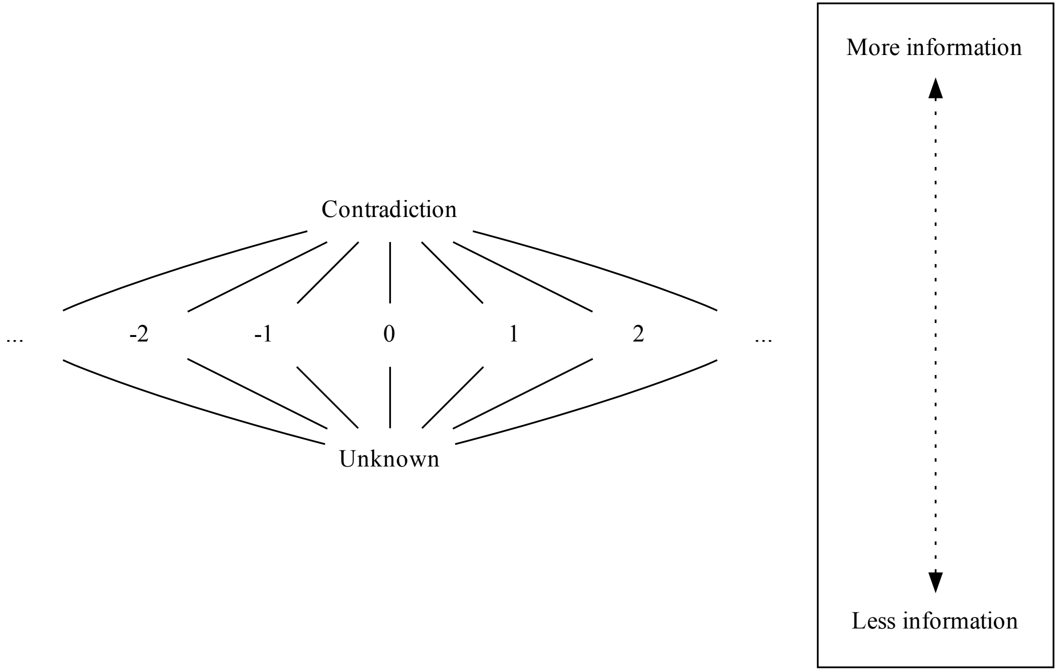

-
3
+
4
-

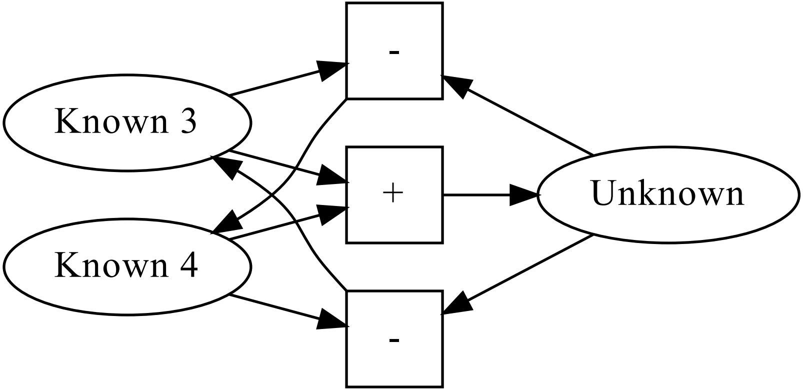
```
data Perhaps a = Unknown | Known a | Contradiction
```

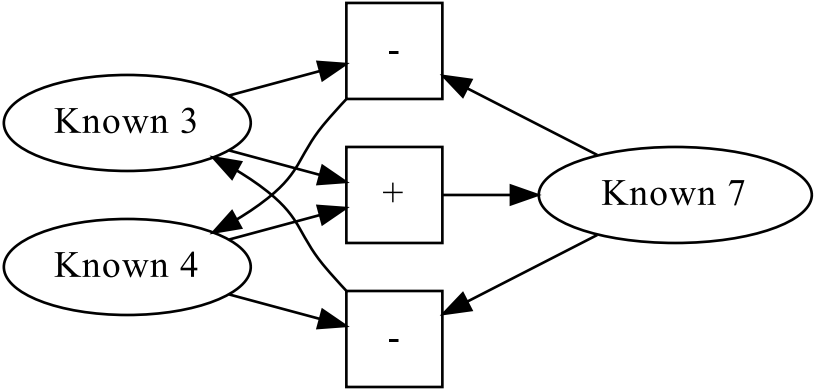
```haskell
data Perhaps a = Unknown | Known a | Contradiction

instance Eq a => BoundedJoinSemiLattice (Perhaps a) where
  bottom = Unknown
  (\/) Unknown x             = x
  (\/) x       Unknown       = x
  (\/) Contradiction _       = Contradiction
  (\/) _       Contradiction = Contradiction
  (\/) (Known a) (Known b) =
    if a == b
      then Known a
      else Contradiction
```

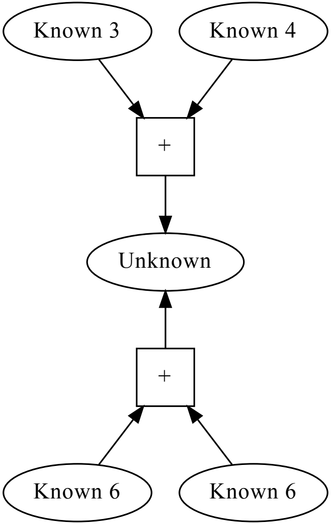

Contradiction
...
-2
-1
0
1
2
...
Unknown

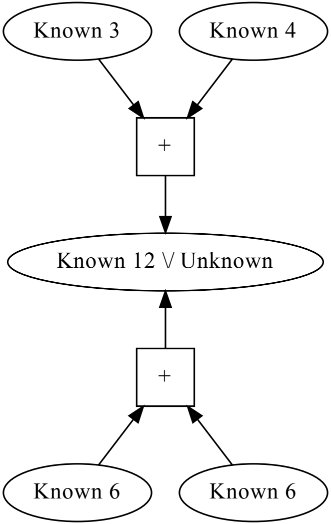

Contradiction
... -2 -1 0 1 2 ...
Unknown
More information
Less information

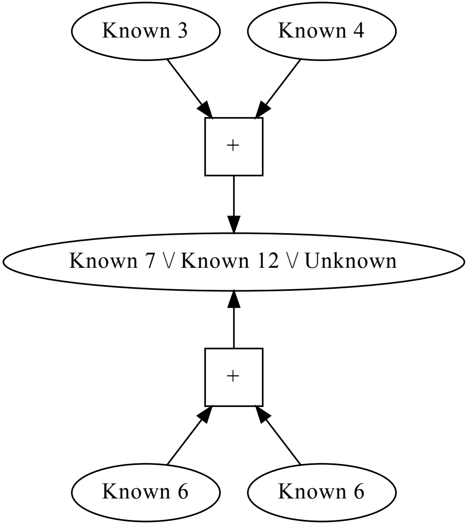

-
Known 3
+
Unknown
Known 4
-

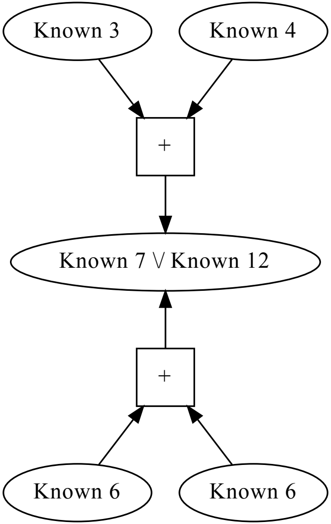


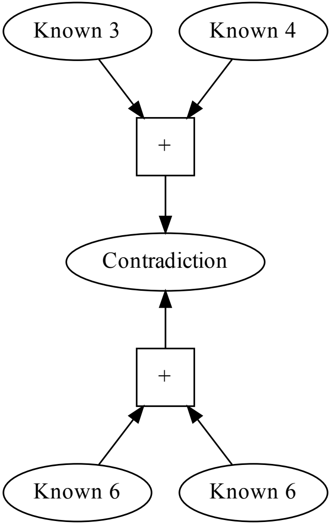
-
Known 3
+
Known 7
Known 4
-


Known 3
Known 4
+
Unknown
+
Known 6
Known 6


Known 3
Known 4
+
Known 12 ∨ Unknown
+
Known 6
Known 6


Known 3
Known 4
+
Known 7 ∨ Known 12 ∨ Unknown
+
Known 6
Known 6


Known 3
Known 4
+
Known 7 \/ Known 12
+
Known 6
Known 6


Known 3
Known 4
+
Contradiction
+
Known 6
Known 6

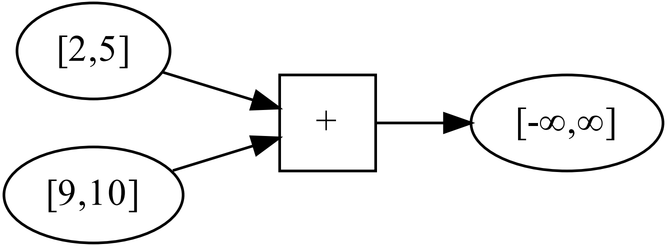
There are loads of other bounded join-semilattices too!

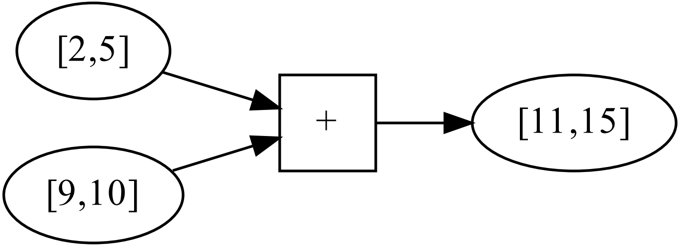
$$[1, 5]$$

$$[1,5] \cap [2,7] = [2,5]$$

$$[1, 5] \cap [2, 7] = [2, 5]$$

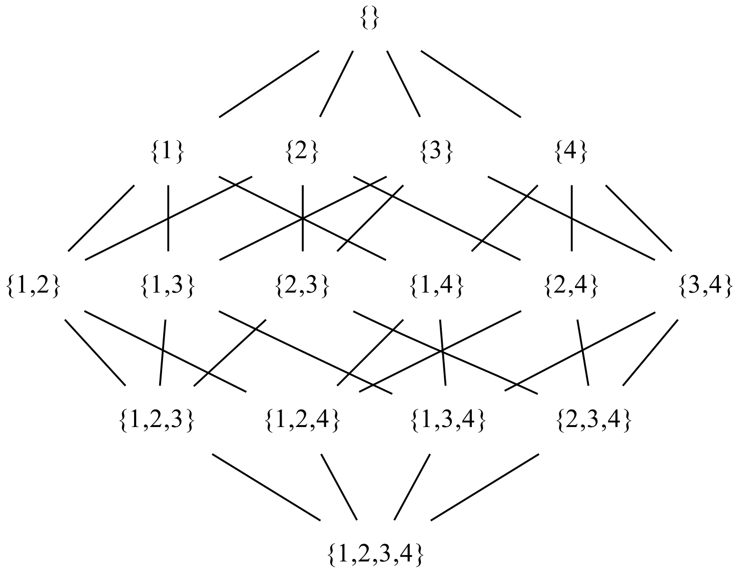
$$[2, 5] + [9, 10] = [11, 15]$$

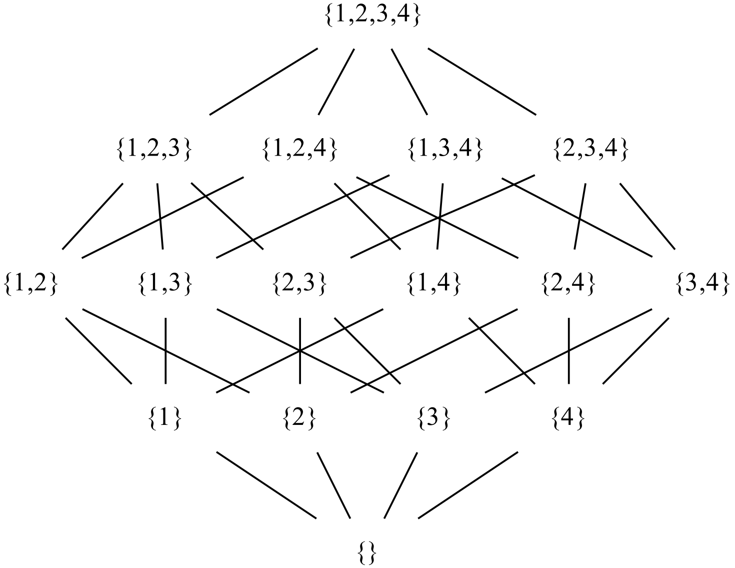

[2,5]
[9,10]
+
[-∞,∞]


[2,5]
[9,10]
+
[11,15]

We can use this to combine multiple imprecise measurements

What other bounded join-semilattices are there?


{}
{1}
{2}
{3}
{4}
{1,2}
{1,3}
{2,3}
{1,4}
{2,4}
{3,4}
{1,2,3}
{1,2,4}
{1,3,4}
{2,3,4}
{1,2,3,4}


{1,2,3,4}
{1,2,3}
{1,2,4}
{1,3,4}
{2,3,4}
{1,2}
{1,3}
{2,3}
{1,4}
{2,4}
{3,4}
{1}
{2}
{3}
{4}
{}

- Set intersection or union
- Interval intersection
- `Perhaps`

And so many more!

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?

What happens when we hit contradiction?

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:(

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Then we can keep track of which subsets of the information are consistent

and which are inconsistent

$$[2, 5] \cap [3, 7] \cap [6, 9] = []$$

$$[2, 5] \cap [3, 7] \cap [6, 9] = []$$

$$[2, 5] \cap [3, 7] = [3, 5]$$

$$[2,5] \cap [3,7] \cap [6,9] = []$$

$$[2,5] \cap [3,7] = [3,5]$$

$$[3,7] \cap [6,9] = [6,7]$$

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$$[2, 5] \cap [3, 7] = [3, 5]$$

$$[3, 7] \cap [6, 9] = [6, 7]$$

$$[2, 5] \cap [6, 9] = []$$

$$[2,5] \cap [3,7] \cap [6,9] = []$$

$$[2,5] \cap [3,7] = [3,5]$$

$$[3,7] \cap [6,9] = [6,7]$$

$$[2,5] \cap [6,9] = []$$

Consistent subsets:

$$\{\}$$
$$\{[2,5]\}$$
$$\{[3,7]\}$$
$$\{[6,9]\}$$
$$\{[2,5],[3,7]\}$$
$$\{[3,7],[6,9]\}$$

$$[2,5] \cap [3,7] \cap [6,9] = []$$

$$[2,5] \cap [3,7] = [3,5]$$

$$[3,7] \cap [6,9] = [6,7]$$

$$[2,5] \cap [6,9] = []$$

Consistent subsets:

$\{\}$
$\{[2,5]\}$
$\{[3,7]\}$
$\{[6,9]\}$
$\{[2,5],[3,7]\}$
$\{[3,7],[6,9]\}$

Maximal consistent subsets:

$\{[2,5],[3,7]\}$
$\{[3,7],[6,9]\}$

$$[2,5] \cap [3,7] \cap [6,9] = []$$

$$[2,5] \cap [3,7] = [3,5]$$

$$[3,7] \cap [6,9] = [6,7]$$

$$[2,5] \cap [6,9] = []$$

Consistent subsets:

$\{\}$
$\{[2,5]\}$
$\{[3,7]\}$
$\{[6,9]\}$
$\{[2,5],[3,7]\}$
$\{[3,7],[6,9]\}$

Inconsistent subsets:

$\{[2,5],[6,9]\}$
$\{[2,5],[3,7],[6,9]\}$

Maximal consistent subsets:

$\{[2,5],[3,7]\}$
$\{[3,7],[6,9]\}$

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$$[3,7] \cap [6,9] = [6,7]$$

$$[2,5] \cap [6,9] = []$$

Consistent subsets:

$\{\}$

$\{[2,5]\}$

$\{[3,7]\}$

$\{[6,9]\}$

$\{[2,5],[3,7]\}$

$\{[3,7],[6,9]\}$

Maximal consistent subsets:

$\{[2,5],[3,7]\}$

$\{[3,7],[6,9]\}$

Inconsistent subsets:

$\{[2,5],[6,9]\}$

$\{[2,5],[3,7],[6,9]\}$

Minimal inconsistent subsets:

$\{[2,5],[6,9]\}$

This concept is something called a *Truth Management System*

Now that we can handle contradiction, we can make guesses!

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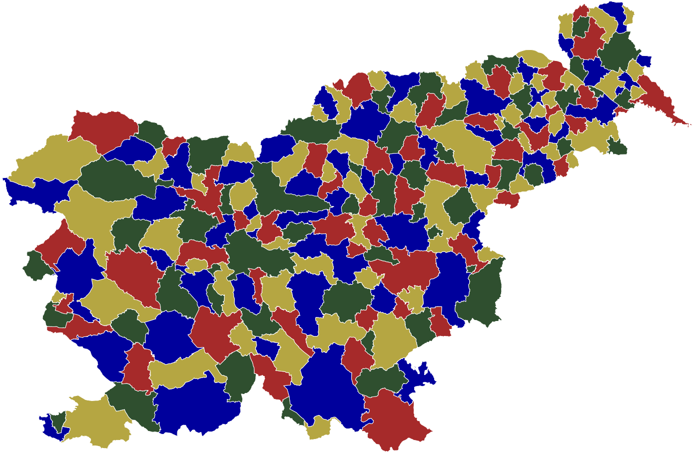
This lets us encode search problems easily



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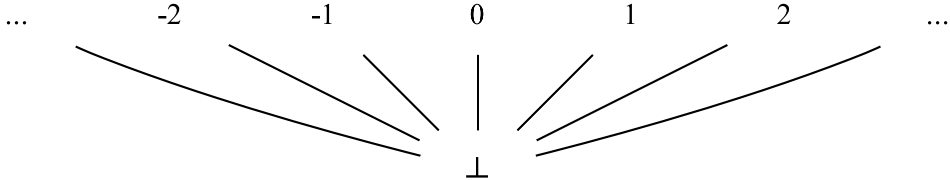
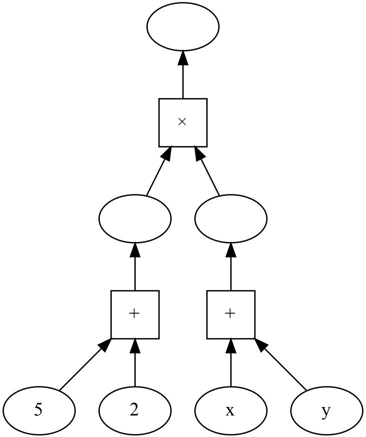
We can relax some of our conditions in certain circumstances


... -2 -1 0 1 2 ...
⊥

We can turn any expression tree into a propagator network
There will only ever be one writer to a cell

$$(5 + 2) \times (x + y)$$



# Wrapping up

Alexey Radul's work on propagators:

- Art of the Propagator
  `http://web.mit.edu/~axch/www/art.pdf`
- Propagation Networks: A Flexible and Expressive Substrate for Computation
  `http://web.mit.edu/~axch/www/phd-thesis.pdf`

Lindsey Kuper's work on LVars is closely related, and works today:

- Lattice-Based Data Structures for Deterministic Parallel and Distributed Programming
  `https://www.cs.indiana.edu/~lkuper/papers/lindsey-kuper-dissertation.pdf`
- lvish library
  `https://hackage.haskell.org/package/lvish`

Edward Kmett has worked on:

- Making propagators go fast
- Scheduling strategies and garbage collection
- Relaxing requirements (Eg. not requiring a full join-semilattice, admitting non-monotone functions)

Ed's stuff:

- `http://github.com/ekmett/propagators`
- `http://github.com/ekmett/concurrent`
- Lambda Jam talk (Normal mode): `https://www.youtube.com/watch?v=acZkF6Q2XKs`
- Boston Haskell talk (Hard mode): `https://www.youtube.com/watch?v=DyPzPeOPgUE`

In conclusion, propagator networks:

- Admit any Haskell function you can write today . . .
- . . . and more functions!
- compute bidirectionally
- give us constraint solving and search
- mix all this stuff together
- parallelise and distribute

Thanks for listening!